

On the circuit diameter of dual transportation polyhedra

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Abstract. In this paper we introduce the circuit diameter of polyhedra, which is always bounded from above by the combinatorial diameter. We consider dual transportation polyhedra defined on general bipartite graphs. For complete $M \times N$ bipartite graphs the Hirsch bound $(M-1)(N-1)$ on the *combinatorial diameter* is a known tight bound (Balinski, 1984). For the *circuit diameter* we show the much stronger bound $M+N-2$ for all dual transportation polyhedra defined on arbitrary bipartite graphs with $M+N$ nodes.

Keywords: augmentation, Graver basis, test set, circuit, elementary vector, linear program, integer program, diameter, Hirsch conjecture

1 Introduction.

Graver bases of matrices $A \in \mathbb{Z}^{d \times n}$ were introduced by Jack Graver in 1975 in his seminal paper [5] as sets $\mathcal{G}(A)$ of vectors that provide optimality certificates for the family of integer linear programs $\min \{ \mathbf{c}^\top \mathbf{z} : A\mathbf{z} = \mathbf{b}, \mathbf{l} \leq \mathbf{z} \leq \mathbf{u}, \mathbf{z} \in \mathbb{Z}^n \}$ that share the problem matrix A but that may differ in the remaining data $\mathbf{b}, \mathbf{c}, \mathbf{l}, \mathbf{u}$. This optimality certificate provided by $\mathcal{G}(A)$ allows to augment any given feasible solution to optimality via a simple scheme similar to the Simplex method for linear programs: iteratively augment the given solution along Graver basis directions until a solution is reached that cannot be augmented along a direction from $\mathcal{G}(A)$. This solution must be optimal. Clearly, the number of augmentation steps needed heavily depends on how one chooses among several applicable augmenting Graver basis directions.

In the last 20 years, a lot of progress has been made on the theory of Graver bases. It has been shown that $\mathcal{G}(A)$ also provides optimality certificates for the minimization of separable convex objective functions over the lattice points of a polyhedron [10], that at most polynomially many (in the binary encoding length of the input data) Graver-best augmentation steps are needed in order to reach an optimal solution [8], and that N -fold separable-convex integer linear programs can be solved in polynomial time [4,6,7]. For a more thorough introduction to the theory of Graver bases and for more references on this topic we refer the interested reader to the books [2,11].

Note that the notion of a Graver basis can be extended to the continuous setting of linear programs. Here, the *circuits* or *elementary vectors* $\mathcal{C}(A)$ of $A \in \mathbb{Z}^{d \times n}$ provide a universal optimality certificate similarly as the Graver basis $\mathcal{G}(A)$ does for the integer setting. All results readily translate from the integer linear to the linear setting and the proofs are often much simpler.

Very recently, it has been shown in [3] that for integer linear programs and for linear programs one needs at most $|\mathcal{G}(A)|$ respectively $|\mathcal{C}(A)|$ many steepest-descent Graver basis augmentation steps. This surprising bound does not depend on \mathbf{b} , \mathbf{c} , \mathbf{l} and \mathbf{u} and readily implies that N -fold (integer) linear programs can be solved in *strongly* polynomial time. This raises the natural question of how many circuit augmentation steps are needed with a “perfect” selection rule? Progress on this question may lead to a strongly polynomial-time algorithm for the solution of general linear programs via circuit augmentations, which would solve a long-standing open question on the complexity of LPs. The search for a best selection rule leads us to a notion similar to the *combinatorial diameter* of a polyhedron, which gives a lower bound for the number of steps needed by the Simplex method to solve an LP.

In this paper, we introduce the notion of *circuit diameter* of a polyhedron as the maximum number of (maximum length) steps along circuit directions that are needed to go from any vertex of the polyhedron to any other vertex of the polyhedron. From the definition of the circuits it will follow directly that the circuit diameter of a polyhedron is bounded from above by the combinatorial diameter and thus it is natural to ask, whether the *Hirsch bound* (which has been disproved to bound the combinatorial diameter in general [9,12]) always bounds the circuit diameter of a polyhedron.

Conjecture 1 (Circuit diameter bound). For any n -dimensional polyhedron with f facets the circuit diameter is bounded above by $f - n$.

It is an immediate interesting open question whether the counterexamples to the Hirsch conjecture [9,12] give rise to counterexamples to our Conjecture 1 or not.

To bound the combinatorial diameter of a polyhedron it suffices to consider generic polyhedra, as by perturbation any polyhedron can be turned into a generic polyhedron, whose diameter is at least as big as the one of the original polyhedron. It is not clear whether the same is true for the circuit diameter, see the second example presented in the next section.

In this paper, we consider dual transportation polyhedra defined on general bipartite graphs. For complete $M \times N$ bipartite graphs the Hirsch bound $(M-1)(N-1)$ on the *combinatorial diameter* has been already proved and shown to be tight [1]. For the *circuit diameter* we show the much stronger bound $M+N-2$ for all dual transportation polyhedra defined on bipartite graphs with $M+N$ nodes. This shows that there are families of polyhedra whose circuit diameter is much smaller than their combinatorial diameter, which gives hope that an augmentation algorithm along circuit directions could have a much better complexity to solve LPs than the Simplex method.

2 Circuit distance and circuit diameter

The *circuits* or *elementary vectors* of a matrix $A \in \mathbb{Z}^{d \times n}$ are the support-minimal elements in $\ker(A) \setminus \{\mathbf{0}\}$, normalized to (coprime) integer components. Clearly, there are only finitely many such vectors. It can be shown that the set of circuits consists exactly of all edge directions of $\{\mathbf{z} : A\mathbf{z} = \mathbf{b}, \mathbf{z} \geq \mathbf{0}\}$ for varying \mathbf{b} [13]. This also implies that the set of circuits of A provides a universal optimality certificate for linear programs $\min \{\mathbf{c}^\top \mathbf{z} : A\mathbf{z} = \mathbf{b}, \mathbf{z} \geq \mathbf{0}\}$ for any choice of \mathbf{b} and \mathbf{c} ; similarly as the Graver basis of A does for the integer setting.

In analogy to this, we define for the linear program $\min \{\mathbf{c}^\top \mathbf{z} : A\mathbf{z} \leq \mathbf{b}\}$ the set of circuits $\mathcal{C}_{\leq}(A)$ as the collection of all edge directions of $\{\mathbf{z} : A\mathbf{z} \leq \mathbf{b}\}$ for varying \mathbf{b} . (Note that the matrix $A \in \mathbb{Z}^{d \times n}$ should have full row rank n for the polytope to have vertices and edges.) It is not hard to show that these edge directions are given by those vectors $\mathbf{z} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for which $\text{supp}(A\mathbf{z})$ is inclusion-minimal among all supports $\text{supp}(A\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. It is also not hard to show that $\mathcal{C}_{\leq}(A)$ provides augmenting directions to any non-optimal solution of $\min \{\mathbf{c}^\top \mathbf{z} : A\mathbf{z} \leq \mathbf{b}\}$ for any choice of \mathbf{b} and \mathbf{c} .

One should note that for a linear program, augmentation along circuit directions is a generalization of the Simplex method: While in the Simplex method one walks only along the 1-skeleton/edges (so in particular on the boundary) of the polyhedron, the circuit steps are allowed to go through the interior of the polyhedron (along *potential* edge directions). While [8] states that there is a selection strategy such that only polynomially many circuit augmentation steps are needed to reach an optimal solution (a fact that is still unresolved for the Simplex method), it is still open how to implement this greedy-type augmentation oracle in polynomial time.

Inspired by the surprising bound of at most $|\mathcal{C}(A)|$ circuit augmentations [3], one may wonder if there is a selection strategy such that only a strongly polynomial number (that depends only on d and n) of augmentation steps is needed to reach an optimal solution. We will not answer this fundamental question here, but introduce and turn to an intimately related problem. For this, let us define the notions of *circuit distance* and *circuit diameter*.

Definition 1. Let $P = \{\mathbf{z} : A\mathbf{z} \leq \mathbf{b}\}$ be a polyhedron. For two vertices $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ of P , we call a sequence $\mathbf{v}^{(1)} = \mathbf{y}^{(0)}, \dots, \mathbf{y}^{(k)} = \mathbf{v}^{(2)}$ a circuit walk of length k if for all $i = 0, \dots, k-1$ we have

1. $\mathbf{y}^{(i)} \in P$,
2. $\mathbf{y}^{(i+1)} - \mathbf{y}^{(i)} = \alpha_i \mathbf{g}^{(i)}$ for some $\mathbf{g}^{(i)} \in \mathcal{C}_{\leq}(A)$ and $\alpha_i > 0$, and
3. $\mathbf{y}^{(i)} + \alpha \mathbf{g}^{(i)}$ is infeasible for all $\alpha > \alpha_i$.

The circuit distance $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})$ from $\mathbf{v}^{(1)}$ to $\mathbf{v}^{(2)}$ then is the minimum length of a circuit walk from $\mathbf{v}^{(1)}$ to $\mathbf{v}^{(2)}$. The circuit diameter $\text{diam}_{\mathcal{C}}(P)$ of P is the maximum circuit distance between any two vertices of P .

It should be noted that a circuit walk is not necessarily reversible, so we may have $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) \neq \text{dist}_{\mathcal{C}}(\mathbf{v}^{(2)}, \mathbf{v}^{(1)})$. The following example demonstrates that this can indeed happen.

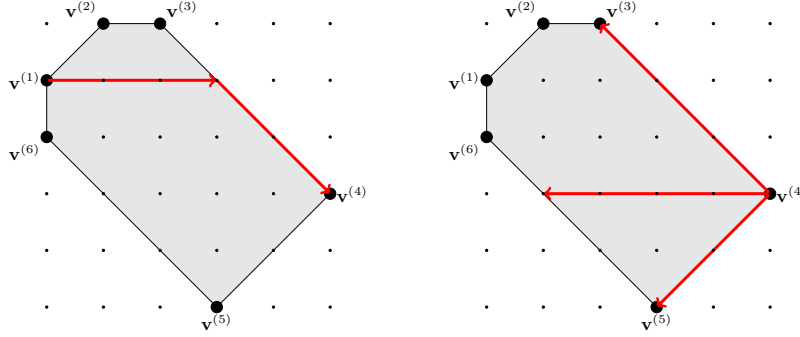
Example 1. Consider the polyhedron $P = \{ \mathbf{z} : A\mathbf{z} \leq \mathbf{b} \}$ given by

$$A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \\ 0 \end{pmatrix}.$$

P is a two-dimensional polytope with six vertices, whose circuits are given by

$$\mathcal{C}_{\leq}(A) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Now let us have a look at the circuit distances $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(1)}, \mathbf{v}^{(4)})$ and $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(4)}, \mathbf{v}^{(1)})$:



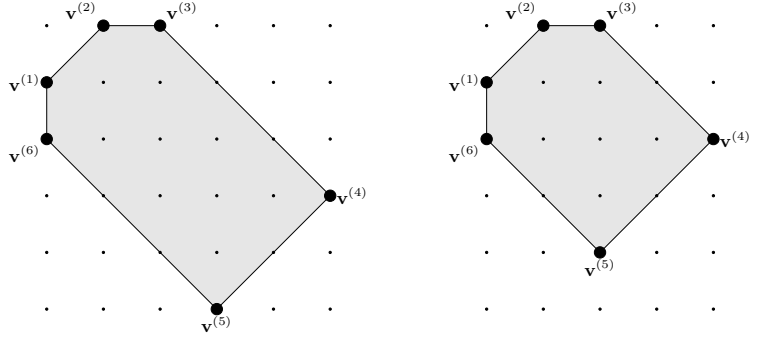
We have $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(1)}, \mathbf{v}^{(4)}) = 2$, but $\text{dist}_{\mathcal{C}}(\mathbf{v}^{(4)}, \mathbf{v}^{(1)}) = 3$. No matter which circuit direction we choose for a first step starting at $\mathbf{v}^{(4)}$, we cannot go to $\mathbf{v}^{(1)}$ with only one more step. \square

The following example demonstrates that perturbing the right-hand side vector may not change the combinatorial structure of the polyhedron while changing the circuit diameter. Note that both polyhedra possess the same set of edge directions/circuits.

Example 2. Consider the polyhedron $\tilde{P} = \{ \mathbf{z} : A\mathbf{z} \leq \tilde{\mathbf{b}} \}$ given by

$$A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{b}} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 4 \\ 0 \end{pmatrix}.$$

P and \tilde{P} have the same combinatorial structure:



It is not hard to check that $\text{diam}_C(P) = 3$ while $\text{diam}_C(\tilde{P}) = 2$. This indicates that perturbing the right-hand side may have effects on the circuit diameter that are hard to predict. \square

Clearly, the circuit diameter of a polyhedron is at most as large as the combinatorial diameter of the polyhedron, as a walk along the 1-skeleton/edges of the polyhedron is a circuit walk. This raises the natural question whether the well-known Hirsch conjecture holds for the circuit diameter in place of the combinatorial diameter. Recall that there are counterexamples to the Hirsch conjecture bounding the combinatorial diameter of polyhedra and polytopes [9,12].

In the following section we consider the circuit diameter of dual transportation polyhedra defined on bipartite graphs $G = (V, E)$ (that are not necessarily complete). We show that their circuit diameter is bounded from above by $|V| - 2$.

3 Dual transportation polyhedra

Let $G = (V, E)$ be a connected bipartite graph on node sets $V_1 = \{0, \dots, M-1\}$ and $V_2 = \{M, \dots, M+N-1\}$ with edges E having one endpoint in V_1 and one endpoint in V_2 . A dual transportation polyhedron associated to G is given by some vector $\mathbf{c} \in \mathbb{R}^{|E|}$ via

$$P_{G,\mathbf{c}} = \{ \mathbf{u} \in \mathbb{R}^{M+N} : -u_a + u_b \leq c_{ab} \ \forall \ a \in V_1, b \in V_2 \text{ and } ab \in E, u_0 = 0 \}.$$

As is standard, we put $u_0 = 0$ to make $P_{G,\mathbf{c}}$ pointed. When we consider the circuit diameter of a specific polyhedron $P_{G,\mathbf{c}}$, we may assume that none of the inequalities $-u_a + u_b \leq c_{ab}$ is redundant (otherwise remove such an edge ab from G , leaving the polyhedron the same but making the set of circuits smaller and thus the circuit diameter potentially bigger).

Moreover, we may assume that $P_{G,\mathbf{c}}$ is generic, although this merely simplifies the presentation. The vertices of $P_{G,\mathbf{c}}$ are determined by sets of inequalities $-u_a + u_b \leq c_{ab}$ that become tight. For $\mathbf{u} \in P_{G,\mathbf{c}}$, we denote by $G(\mathbf{u})$ the graph with nodes V and with edges $ab \in E$ for which $-u_a + u_b \leq c_{ab}$ is tight. For a vertex \mathbf{u} of $P_{G,\mathbf{c}}$, $G(\mathbf{u})$ is a spanning subgraph of G which is always a spanning tree of G if $P_{G,\mathbf{c}}$ is generic. (This can be proved on similar lines as in [1] for the complete bipartite graph.) For our proofs it will be enough to know that for each vertex of $P_{G,\mathbf{c}}$ there is a spanning tree of G with edges corresponding to the inequalities $-u_a + u_b \leq c_{ab}$ that are tight at the vertex. This uniquely determines the vertex \mathbf{u} , since we normalized $u_0 = 0$.

As in [1], the possible edge directions of $P_{G,\mathbf{c}}$ can be described as follows: Let $R, S \subseteq V$ be connected nonempty node sets with $R \cup S = V$ and $R \cap S = \emptyset$. W.l.o.g., we may assume $0 \in R$. Then the vector $\mathbf{g} \in \mathbb{R}^{M+N}$ with

$$g_i = \begin{cases} 0, & \text{if } i \in R, \\ 1, & \text{if } i \in S, \end{cases} \quad (1)$$

is an edge direction of $P_{G,\mathbf{c}}$ for some right-hand side \mathbf{c} . In fact, it can be shown that these are all possible edge directions and hence they constitute the set of circuits, \mathcal{C}_G , associated to the matrix defining the polyhedron $P_{G,\mathbf{c}}$.

We are ready to present and prove the core part of our main result.

Lemma 1. *The circuit diameter $\text{diam}_{\mathcal{C}}(P_{G,\mathbf{c}})$ of $P_{G,\mathbf{c}}$ is bounded from above by $|V| - 1$.*

Proof. Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be two vertices of $P_{G,\mathbf{c}}$ given by the spanning trees $T_1 = G(\mathbf{u}^{(1)})$ and $T_2 = G(\mathbf{u}^{(2)})$ of G . We will show how to construct a circuit walk $\mathbf{u}^{(1)} = \mathbf{y}^{(0)}, \dots, \mathbf{y}^{(k)} = \mathbf{u}^{(2)}$, such that $G(\mathbf{y}^{(i)})$ has at least i edges in common with T_2 . This immediately implies $k \leq |V| - 1$ which proves the claim.

It should be noted that the subgraphs $G(\mathbf{y}^{(i)})$ may not be connected, since our circuit walk possibly goes through the interior of $P_{G,\mathbf{c}}$ along (potential) edge directions and thus may enter the interior of higher-dimensional faces of $P_{G,\mathbf{c}}$.

Given the feasible point $\mathbf{y}^{(i)} \neq \mathbf{u}^{(2)}$, let $C = G(V(C), E(C))$ be the connected component of $(V, E(G(\mathbf{y}^{(i)})) \cap E(T_2))$ containing the node 0. Possibly, C consists only of the node 0. As $\mathbf{y}^{(i)} \neq \mathbf{u}^{(2)}$, we must have $C \neq T_2$ and thus there is some node $s \in V$ which is not in C , but which is connected to C directly via some edge rs in T_2 . We now construct an edge direction \mathbf{g} from \mathcal{C}_G such that $\mathbf{y}^{(i+1)} := \mathbf{y}^{(i)} + \alpha \mathbf{g}$ arises from a maximal length step along \mathbf{g} and such that $(V, E(G(\mathbf{y}^{(i+1)})) \cap E(T_2))$ contains C and the edge rs from T_2 . Starting from $\mathbf{y}^{(0)}$ and repeating this process iteratively, we see that $G(\mathbf{y}^{(i)})$ has at least i edges in common with T_2 , implying the result.

To construct \mathbf{g} , we need to define $R, S \subseteq V$ that describe the edge direction from \mathcal{C}_G . W.l.o.g. we will assume that $s \in V_2$. The case $s \in V_1$ works analogously by merely switching the roles of V_1 and V_2 and hence by switching the roles of $\epsilon \mathbf{g}$ and $-\epsilon \mathbf{g}$ below.

(a) All nodes from C are assigned to R .

- (b) All nodes from $V_2 \setminus \{s\}$ which are connected to C by an edge in E , are assigned to R .
- (c) All nodes $t \in V \setminus R$ that are connected to s by a path in G are assigned to S .
- (d) All remaining nodes are assigned to R .

As G is connected, this construction leads to sets R and S that are nonempty and that define connected components of G that are connected by an edge $rs \in E$. Hence, R and S define an element $\mathbf{g} \in \mathcal{C}_G$ via Equation (1). We wish to include the edge rs into our graph, that is, we wish to make the inequality $-u_r + u_s \leq c_{rs}$ tight at $\mathbf{y}^{(i+1)}$, that is, we wish to increase the component $y_s^{(i)}$ ($s \in V_2$ and as $s \notin V(C)$). Hence we *add* $\epsilon \mathbf{g}$ to $\mathbf{y}^{(i)}$. (If $s \in V_1$, we *subtract* $\epsilon \mathbf{g}$ from $\mathbf{y}^{(i)}$.) We choose as ϵ the smallest nonnegative number such that an inequality $-u_a + u_b \leq c_{ab}$ with $a \in R$ and $b \in S$ becomes tight. Note that $\epsilon = 0$ is not excluded, but we show that this will never happen. In fact, we show that the edge ab (on which $-u_a + u_b \leq c_{ab}$ becomes tight) is exactly the edge rs that we wish to include.

Assume now on the contrary that $ab \neq rs$. Note that by construction at steps (b) and (c) we must have $b = s$, as all edges from R to $S \cap V_2$ have s as common end point and these are exactly the edges on which an inequality may become tight when walking along direction $\mathbf{g} \in \mathcal{C}_G$. Hence we must have $a \neq r$. As $G(\mathbf{y}^{(i)} + \epsilon \mathbf{g})$ and T_2 coincide on the edges in C and since $0 \in V(C)$, $\mathbf{y}^{(i+1)} := \mathbf{y}^{(i)} + \epsilon \mathbf{g}$ and $\mathbf{u}^{(2)}$ agree in their components in $V(C)$, that is, $u_c^{(2)} = y_c^{(i+1)}$ for all $c \in V(C)$. Since $\mathbf{y}^{(i+1)} \in P_{G,c}$ and since $as \in E(G(\mathbf{y}^{(i+1)}))$ and $rs \notin E(G(\mathbf{y}^{(i+1)}))$, we have

$$-u_a^{(2)} + y_s^{(i+1)} = c_{as} \text{ but } -u_r^{(2)} + y_s^{(i+1)} < c_{rs}.$$

On the other hand, since $\mathbf{u}^{(2)} \in P_{G,c}$ and since $as \notin E(T_2)$ and $rs \in E(T_2)$, we have

$$-u_a^{(2)} + u_s^{(2)} < c_{as} \text{ but } -u_r^{(2)} + u_s^{(2)} = c_{rs}.$$

From $-u_a^{(2)} + y_s^{(i+1)} = c_{as}$ and $-u_a^{(2)} + u_s^{(2)} < c_{as}$ we conclude $y_s^{(i+1)} > u_s^{(2)}$, whereas $-u_r^{(2)} + y_s^{(i+1)} < c_{rs}$ and $-u_r^{(2)} + u_s^{(2)} = c_{rs}$ imply $y_s^{(i+1)} < u_s^{(2)}$. This contradiction shows $a = r$ and the claim is proved. \square

We can strengthen this result by observing the following fact on the vertices of $P_{G,c}$.

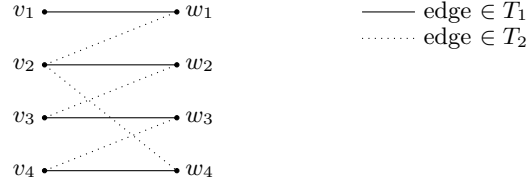
Lemma 2. *Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be two vertices of $P_{G,c}$ given by the spanning trees T_1 and T_2 of G . Then $E(T_1) \cap E(T_2) \neq \emptyset$.*

Proof. As we can translate $P_{G,c}$, we may assume w.l.o.g. that $\mathbf{u}^{(1)} = \mathbf{0}$. Clearly, this mere shift does not change any structure, in particular, it does not change $T_1 = G(\mathbf{u}^{(1)})$ and $T_2 = G(\mathbf{u}^{(2)})$. For better readability, let us denote the components of $\mathbf{u}^{(i)}$, $i = 1, 2$, belonging to V_1 and V_2 , respectively, by $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$. Now assume that $E(T_1) \cap E(T_2) = \emptyset$.

As T_1 is connected, there must be an edge $(v_1, w_1) \in T_1$. As $(v_1, w_1) \notin T_2$, we have $-v_1^{(2)} + w_1^{(2)} < -v_1^{(1)} + w_1^{(1)} = 0$ and hence $w_1^{(2)} < v_1^{(2)}$.

As T_2 is connected, there must be an edge $(v_2, w_1) \in T_2$. As $(v_2, w_1) \notin T_1$, we must have $0 = -v_2^{(1)} + w_1^{(1)} < -v_2^{(2)} + w_1^{(2)}$ and hence $v_2^{(2)} < w_1^{(2)}$.

Again, as T_1 is connected, there must be an edge $(v_2, w_2) \in T_1$. As $(v_2, w_2) \notin T_2$, we have $-v_2^{(2)} + w_2^{(2)} < -v_2^{(1)} + w_2^{(1)} = 0$ and hence $w_2^{(2)} < v_2^{(2)}$.



Continuing like this, we create a path with edges alternately from $T_1 \setminus T_2$ and $T_2 \setminus T_1$. As there are only finitely many nodes, eventually some v_i (or w_j) is selected a second time and we close a cycle. But then we have that

$$v_i^{(2)} > w_i^{(2)} > v_{i+1}^{(2)} > \dots > v_k^{(2)} = v_i^{(2)}$$

(or $w_j^{(2)} > \dots > w_j^{(2)}$), a contradiction. Hence we must have $E(T_1) \cap E(T_2) \neq \emptyset$. \square

This now implies the following strengthening of Lemma 1.

Theorem 1. *The circuit diameter $\text{diam}_C(P_{G,\mathbf{c}})$ of $P_{G,\mathbf{c}}$ is bounded from above by $|V| - 2$.*

Proof. The proof is analogous to the proof of Lemma 1. We merely have to observe that w.l.o.g. we may assume that the edge that is common to $G(\mathbf{u}^{(1)})$ and to $G(\mathbf{u}^{(2)})$ has 0 as one of its endpoints. So we only have to add at most $|V| - 2$ edges in at most $|V| - 2$ steps. \square

Acknowledgments

The second author gratefully acknowledges the support from the graduate program TopMath of the Elite Network of Bavaria and the TopMath Graduate Center of TUM Graduate School at Technische Universität München.

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